SMARANDACHE DISJOINT IN BCK/D-ALGEBRAS

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Received September 24, 2004

ABSTRACT. In this paper we include several new families of Smarandache-type P-algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

Let $(X,*)$ be a binary system/algebra. Then $(X,*)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y,*)$, where Y is non-trivial, i.e., $|Y| \geq 2$, or Y contains at least two distinct elements, and $(Y,*)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type P-algebra is an abelian group). Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [2]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

In this paper we include several new families of Smarandache-type P-algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

A *d-algebra* ([4]) $(X,*,0)$ is an algebra satisfying the following axioms: (i) $x*x=0$ for all $x \in X$; (ii) $0*x=0$ for all $x \in X$; (iii) $x*y=y*x=0$ if and only if $x=y$.

If $X=[0,\infty)=\{x \in R \mid x \geq 0\}$, where R is the collection of real numbers, and if $x*y=\max\{0,x-y\}$, then $(X,*,0)$ is a d-algebra. d-algebras are quite common and occur in many situations as the example above indicates.

Instead of asking whether a d-algebra can be a semigroup using the same operation, we can instead ask the much wider question: Can a d-algebra be a Smarandache-type semigroup ?

Theorem 1. *If $(X,*,0)$ is a d-algebra, then it cannot be a Smarandache-type semigroup.*

Proof. Suppose that it is in fact a Smarandache-type semigroup. Let $|Y| \geq 2$, where $(Y,*)$ is a subalgebra of $(X,*)$, which is also a semigroup. Thus, if $y \in Y$, then $y*y=0 \in Y$ as well. Therefore $(y*y)*y=0*y=0=y*(y*y)=y*0$. But $y*0=0*y=0$ by condition (iii) for d-algebras implies $y=0$, so that in fact $Y=\{0\}$ and $|Y|=1$, a contradiction. □

Corollary 2. *If $(X,*,0)$ is a d-algebra, then it cannot be a Smarandache-type group.*

We can say the following however:

Theorem 3. *If $(X,*)$ is a semigroup, then it cannot be a Smarandache-type d-algebra.*

2000 *Mathematics Subject Classification.* 06F35.
Key words and phrases. BCK/d-algebras, Smarandache-type, Smarandache disjoint.

Proof. Suppose that $(Y,*,0)$ is a non-trivial sub-d-algebra of $(X,*)$. Then if $y \in Y$, we have $y*y=0$ and $0 \in Y$ and $(y*y)*y = 0*y = 0 = y*(y*y) = y*0$, so that $y=0$ and $|Y|=1$, $Y=\{0\}$. Hence, the condition $|Y| \geq 2$ is impossible, i.e., $(X,*)$ is not a Smarandache-type d-algebra. □

Given a d-algebra and abelian group, we can construct an algebra which is both a Smarandache-type d-algebra and a Smarandache-type abelian group. Let $(Y,*,0)$ be a d-algebra and $(Z,+)$ be the additive abelian group of natural numbers. Let $(X,\otimes)$ be defined for $X = Y \times Z = \{(a,m) \,|\, a \in Y, m \in Z\}$ as follows: $(a,m)\otimes(b,n) := (a*b, m+n)$. Then $(a,0)\otimes(b,0) = (a*b,0)$, and thus $(Y\times\{0\},\otimes,(0,0))$ is a subalgebra of $(X,\otimes)$ which is isomorphic to the d-algebra $(Y,*,0)$. On the other hand, $(\{0\}\times Z,\otimes)$ has a product $(0,m)\otimes(0,n) = (0*0, m+n) = (0,m+n)$ so that $(\{0\}\times Z,\otimes)$ is an example of an algebra which is a Smarandache-type abelian group. Hence $(X,*)$ is both a Smarandache-type d-algebra and Smarandache-type abelian group.

Given algebra types $(X,*)$ (type-P_1) and $(X,\circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* if the following two conditions hold:

(A) If $(X,*)$ is a type-P_1-algebra with $|X|>1$ then it cannot be a Smarandache-type-P_2-algebra $(X,\circ)$;

(B) If $(X,\circ)$ is a type-P_2-algebra with $|X|>1$ then it cannot be a Smarandache-type-P_1-algebra $(X,*)$.

This condition does not exclude the existence of algebras $(X,\diamond)$ which are both Smarandache-type-P_1-algebras and Smarandache-type-P_2-algebras. In fact we have already produced an example of such an algebra where $P_1 \equiv$ 'semigroup' and $P_2 \equiv$ 'd-algebra'.

If $(X,*,0)$ is a d-algebra which also satisfy the following conditions:

(iv) $(x*(x*y))*y = 0$ for all $x,y \in X$;

(v) $((x*y)*(x*z))*(z*y) = 0$ for all $x,y,z \in X$,

then it is a BCK-algebra (see [1, 3]). Since BCK-algebras are d-algebras it follows that:

Theorem 4. *Semigroups and d-algebras are Smarandache disjoint.*

Corollary 5. *Semigroups and BCK-algebras are Smarandache disjoint.*

Corollary 6. *Groups and d-algebras are Smarandache disjoint.*

Of course, since groups are semigroups we have:

Corollary 7. *Groups and semigroups are not Smarandache disjoint.*

Consider the collection of left semigroups, i.e., semigroups $(X,*)$ with an associative product $x*y=x$. If $(Y,*)$ is a subgroup of $(X,*)$, then $x*y=x$ means that y is the multiplicative identity of $(Y,*)$ and since $y \in Y$ is arbitrary, it follows that $|Y|=1$ as well. Hence we show that:

Theorem 8. *If $(X,*)$ is a left semigroup, then $(X,*)$ cannot be a Smarandache-type group.*

If $(X,*,e)$ is a group and if $(Y,*)$ is a left semigroup, then being closed, we have for $x,y \in Y$, $x*y=x$, whence from the group structure of $(X,*)$ we find $y=e$, and since $y \in Y$

is arbitrary, $|Y| = 1$ as well. Thus $(X,*)$ cannot be a Smarandache-type left semigroup and hence we have shown that:

Theorem 9. *Left semigroups and groups are Smarandache disjoint.*

Corollary 10. *If $(X,*)$ with $x * y = y$ is a right semigroup, then it follows that right semigroups and groups are Smarandache disjoint.*

The notion of Smarandache disjointness illustrated here appears to be novel and of interest as well.

Question. *Give an examples of a special class of d-algebras which is Smarandache disjoint from the class of BCK-algebras.*

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